Stochastic resonance with spatiotemporal signal controlled by time delays

A. Krawiecki[1,2] and T. Stemler[2]
[1]*Faculty of Physics and Center of Excellence for Complex Systems Research, Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland*
[2]*Institut für Festkörperphysik, Technische Universität Darmstadt, Hochschulstrasse 6, D-64289 Darmstadt, Germany*
(Received 27 May 2003; published 17 December 2003)

Stochastic resonance in two coupled threshold elements with input periodic signals shifted in phase is studied. For fixed phase shift and coupling strength the signal-to-noise ratio at the output of each element can be maximized by introducing proper time delays in the coupling terms which cancel the effect of the phase shift. This shows that in systems of coupled elements driven by spatiotemporal periodic signals stochastic resonance can be controlled by delayed coupling.

DOI: 10.1103/PhysRevE.68.061101 PACS number(s): 05.40.−a, 05.45.Ra

Stochastic resonance (SR) is a phenomenon occurring in systems driven by a combination of a periodic signal and noise, in which the periodic component of a suitably defined output signal can be maximized with respect to the noisy background for nonzero input noise intensity [1–4]. Typically, SR is characterized by a maximum of the output signal-to-noise ratio (SNR) R appearing for optimum noise. In systems of coupled units, each exhibiting SR, which are driven by identical periodic signals, SR can be usually enhanced due to proper coupling, i.e., the maximum SNR can be increased with respect to that of a single unit [5–9]. In contrast, the enhancement of SR when the coupled units are driven by different periodic signals [10–16] is a more complex problem. For example, in coupled [12] or spatially extended [13,14] systems where the coupled units are driven by periodic phase-shifted signals, SR can be enhanced due to coupling, but the increase of the maximum SNR is less pronounced for a wide range of the coupling strength. In this paper we show that this undesirable effect of the phase shift can be canceled, and the maximum SNR can be increased by introducing proper time delays in the coupling between units. Various methods to enhance SR by means other than varying the noise strength are known to be under the name of controlling SR [17–19]. Hence this paper extends the investigation of the role of time delays in systems with SR [20–25] to the case of controlling SR in systems of coupled units driven by periodic signals shifted in phase.

As a model we use two coupled threshold elements, denoted as $i=1,2$, with discrete time $n=0,1,2,\ldots$, which are driven by harmonic signals with identical amplitudes A and frequencies $\omega_s=2\pi/T_s$, shifted in phase by $\Delta\phi$, $0\leqslant\Delta\phi\leqslant\pi$, and by independent white Gaussian noises $\eta_n^{(i)}$, $i=1,2$, whose intensities are given by their variance D,

$$y_{n+1}^{(1)}=\Theta(A\sin(\omega_s n)+\eta_n^{(1)}+wy_{n-\tau_2}^{(2)}-b),$$

$$y_{n+1}^{(2)}=\Theta(A\sin(\omega_s n+\Delta\phi)+\eta_n^{(2)}+wy_{n-\tau_1}^{(1)}-b). \qquad (1)$$

In the above equation Θ denotes the Heaviside step function, w is the coupling strength, τ_1 and τ_2 are time delays in the coupling term, so that the state of each element influences that of the opposite element only after certain time, and b is the threshold. The output of each element $y_n^{(i)}$ at time n is one if the sum of the input periodic signal, noise, and the contribution from the coupled element exceeds the threshold b, and zero otherwise. As a measure of SR in each element i we take the SNR (in decibel) defined by $R^{(i)}=10\log_{10}S_P^{(i)}(\omega_s)/S_N^{(i)}(\omega_s)$, where the numerator is the height of the peak in the power spectrum density of the time series $y_n^{(i)}$ at the frequency ω_s, and the denominator is the height of the noisy background in the power spectrum density close to ω_s.

It is known that uncoupled threshold elements with $A<b$ exhibit SR [26]; moreover, SR and many related phenomena in coupled systems can be modeled using coupled threshold elements [12,14,22,23,27]. The model in Eq. (1) with $\tau_1=\tau_2=0$ was studied in Ref. [12], where it was shown that for $0\leqslant\Delta\phi<\pi/2$, SR in each element can be enhanced by optimum positive coupling $w>0$, while for $\pi/2<\Delta\phi\leqslant\pi$, SR can be enhanced by negative coupling $w<0$. The coupling between elements in Eq. (1) is typical for artificial neural networks, and it is known that threshold elements can be used as qualitative models for SR in biological neurons [28]. In the latter case time delays can naturally appear in the system due to finite propagation time of action potentials through synaptic connections.

Numerical simulations of Eq. (1) show that SR in each element is observed for a wide range of w, τ_1 and τ_2, i.e., the curves $R^{(i)}$ vs D show maxima at $D>0$ (Fig. 1). Figure 1 also shows strong dependence of the SNR on τ_1 and τ_2. This is further illustrated in Fig. 2 for $w>0$ and in Fig. 3 for $w<0$. For fixed w, optimum combination of delays can significantly increase the maximum SNR compared with the case without delays. By inspection of Figs. 2 and 3 the optimum delays can be found as $\tau_1=T_s-\Delta\phi/\omega_s$, $\tau_2=\Delta\phi/\omega_s$ for $w>0$, and $\tau_1=T_s/2-\Delta\phi/\omega_s$, $\tau_2=T_s/2+\Delta\phi/\omega_s$ for $w<0$ (modulo T_s).

The above results can be understood qualitatively as follows (cf. Refs. [8,12]). Without coupling, the probability to have "one" at the output for each element follows the corresponding input periodic signal. Without time delays, if $w>0$ and $\Delta\phi=0$, the output signal from each element becomes more periodic, since at the input the contribution from the coupled element cooperates with the periodic signal, and hence the SNR increases. Similar cooperation takes place

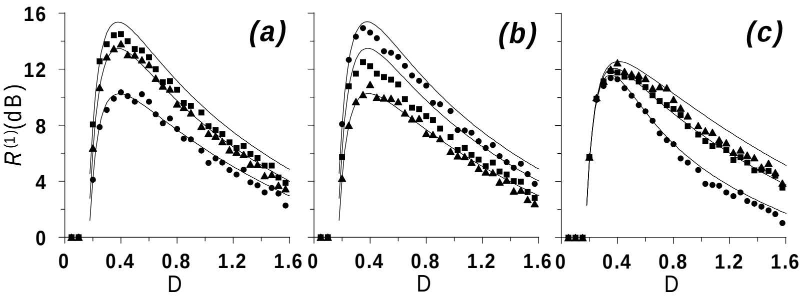

FIG. 1. Numerical (symbols) and corresponding theoretical (solid lines) $R^{(1)}$ vs D for $A=0.1$, $T_s=128$, $\tau_1+\tau_2=T_s$, $b=0.6$, and (a) $w=0.45$, $\Delta\phi=0$, $\tau_1=0$ (squares; optimum delays), $\tau_1=T_s/4$ (triangles), $\tau_1=T_s/2$ (dots); (b) $w=0.45$, $\Delta\phi=\pi/2$, $\tau_1=0$ (squares), $\tau_1=T_s/4$ (triangles), $\tau_1=3T_s/4$ (dots; optimum delays); (c) $w=-1.0$, $\Delta\phi=\pi/2$, $\tau_1=0$ (squares), $\tau_1=T_s/4$ (triangles; optimum delays), $\tau_1=3T_s/4$ (dots).

also for $w>0$ and $0\leqslant\Delta\phi<\pi/2$, though obviously the rise of the SNR is then smaller. On the other hand, if $w<0$ and $\Delta\phi=\pi$, the SNR can also slightly increase because the contribution from the coupled element decreases the probability to have undesirable "one" at the output when the periodic signal is minimum. The situation is similar also for $w<0$ and $\pi/2<\Delta\phi\leqslant\pi$, but, again, the increase of the SNR is smaller. Introducing nonzero delays amounts to replacing $\Delta\phi$ by effective phase shifts $\Delta\phi_i$, perceived by each element, between the phase of the periodic signal at the input of a given element and that at the input of the coupled element i, shifted in time by τ_i, i.e., $\Delta\phi_1=-\Delta\phi-\omega_s\tau_1$, $\Delta\phi_2=\Delta\phi-\omega_s\tau_2$. It follows from the discussion above that the best enhancement of SR should occur when both effective phase shifts have optimum values: $2k\pi$, $k=0,\pm1,\pm2,\ldots$, for $w>0$ and $(2k+1)\pi$ for $w<0$. This condition, together with assumption that $0\leqslant\tau_1,\tau_2\leqslant T_s$, yields the optimum values of τ_1 and τ_2 given above, as can be seen from Figs. 2 and 3.

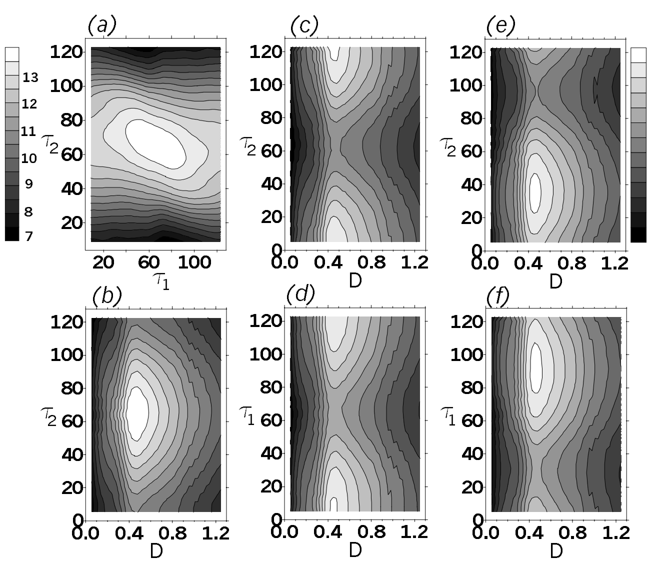

FIG. 2. (a) Contour plots of the maximum $R^{(1)}$ (in decibel) vs τ_1 and τ_2 for $A=0.1$, $T_s=128$, $w=0.45$, $\Delta\phi=\pi$, $b=0.6$, gray scale on the left; (b)–(f) contour plots of the $R^{(1)}$ (in decibel) vs D and τ_1 or τ_2 for $A=0.1$, $T_s=128$, $w=0.45$, $b=0.6$, and (b) $\Delta\phi=\pi$, $\tau_1=T_s/2$; (c) $\Delta\phi=0$, $\tau_1=T_s$; (d) $\Delta\phi=0$, $\tau_1+\tau_2=T_s$; (e) $\Delta\phi=\pi/2$, $\tau_1=3T_s/4$; (f) $\Delta\phi=\pi/2$, $\tau_1+\tau_2=T_s$; gray scale for (b)–(f) on the right.

For a theoretical evaluation of the $R^{(i)}$ we limit ourselves to the case $\tau_1+\tau_2=T_s$ (the optimum delays are located on this line) and to the adiabatic limit of slow periodic signals with $\omega_s\to0$. The stochastic processes $y_n^{(i)}$, $i=1,2$, are cyclostationary, thus the probability that $y_n^{(i)}=1$, denoted as $\mathrm{Prob}(y_n^{(i)}=1)$, is a periodic function of time with period T_s. From this probability $R^{(i)}$ can be obtained as [26]

$$R^{(i)}=10\,\log_{10}\frac{|P_1^{(i)}|^2}{\overline{\mathrm{Prob}(y_n^{(i)}=1)}-\overline{\mathrm{Prob}^2(y_n^{(i)}=1)}},\qquad(2)$$

where $P_1^{(i)}=T_s^{-1}\Sigma_{n=0}^{T_s-1}\mathrm{Prob}(y_n^{(i)}=1)\exp(-i\omega_s n)$ is the Fourier coefficient of the periodic in time probability $\mathrm{Prob}(y_n^{(i)}=1)$ at frequency ω_s, and the bar denotes the time average over T_s. To obtain $\mathrm{Prob}(y_n^{(i)}=1)$ we can write in the Markovian approximation

$$\begin{aligned}\mathrm{Prob}(y_{n+1}^{(1)}=1)&=\mathrm{Prob}(y_{n+1}^{(1)}=1|y_{n-\tau_2}^{(2)}=1)\mathrm{Prob}(y_{n-\tau_2}^{(2)}=1)\\&\quad+\mathrm{Prob}(y_{n+1}^{(1)}=1|y_{n-\tau_2}^{(2)}=0)\\&\quad\times[1-\mathrm{Prob}(y_{n-\tau_2}^{(2)}=1)],\end{aligned}$$

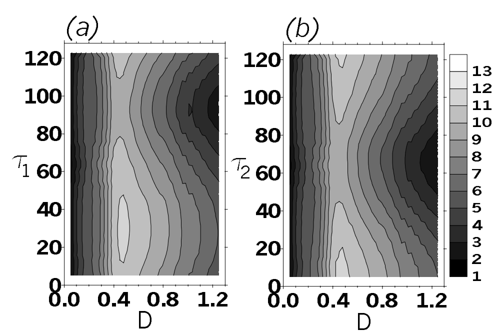

FIG. 3. Contour plots of the $R^{(1)}$ (in decibel) vs D and τ_1 or τ_2, for $A=0.1$, $T_s=128$, $w=-1.0$, $b=0.6$, and (a) $\Delta\phi=\pi/2$, $\tau_1+\tau_2=T_s$; (b) $\Delta\phi=\pi$, $\tau_1=T_s/2$ (nonoptimum), gray scale on the right.

$$\begin{aligned}\mathrm{Prob}(y^{(2)}_{n+1-\tau_2}=1)=&\mathrm{Prob}(y^{(2)}_{n+1-\tau_2}=1|y^{(1)}_{n-\tau_2-\tau_1}=1)\\&\times\mathrm{Prob}(y^{(1)}_{n-\tau_2-\tau_1}=1)\\&+\mathrm{Prob}(y^{(2)}_{n+1-\tau_2}=1|y^{(1)}_{n-\tau_2-\tau_1}=0)\\&\times[1-\mathrm{Prob}(y^{(1)}_{n-\tau_2-\tau_1}=1)],\end{aligned}\tag{3}$$

and a complementary system of equations for $\mathrm{Prob}(y^{(2)}_{n+1}=1)$, $\mathrm{Prob}(y^{(1)}_{n+1-\tau_1}=1)$. The conditional probabilities can be evaluated directly from Eq. (1), e.g.,

$$\mathrm{Prob}(y^{(2)}_{n+1-\tau_2}=1|y^{(1)}_{n-\tau_2-\tau_1}=\beta)=\frac{1}{2}\left\{1-\mathrm{erf}\left[\frac{b-A\sin(\omega_s n+\Delta\phi_2)-w\,\delta_{\beta,1}}{\sqrt{2D^2}}\right]\right\},\tag{4}$$

where $\beta\in\{0,1\}$, $\Delta\phi_2=\Delta\phi-\omega_s\tau_2$ is the effective phase shift, and $\delta_{p,q}$ denotes the Kronecker delta. For $\tau_1+\tau_2=T_s$ there is $\mathrm{Prob}(y^{(1)}_{n-\tau_2-\tau_1}=1)=\mathrm{Prob}(y^{(1)}_n=1)$, and in the adiabatic limit $\mathrm{Prob}(y^{(1)}_{n+1}=1)=\mathrm{Prob}(y^{(1)}_n=1)$, $\mathrm{Prob}(y^{(2)}_{n+1-\tau_2}=1)=\mathrm{Prob}(y^{(2)}_{n-\tau_2}=1)$ can be assumed. Under these assumptions Eq. (3) becomes a closed system of linear equations, and

$$\mathrm{Prob}(y^{(1)}_n=1)=\frac{\alpha_n^{(1,1)}+(\alpha_n^{(2,1)}-\alpha_n^{(1,1)})\alpha_n^{(1,2)}}{1-(\alpha_n^{(2,1)}-\alpha_n^{(1,1)})(\alpha_n^{(2,2)}-\alpha_n^{(1,2)})},\tag{5}$$

where

$$\alpha_n^{(k,l)}=\frac{1}{2}\left\{1-\mathrm{erf}\left[\frac{b-A\sin(\omega_s n+\Delta\phi_2\,\delta_{l,2})-w\,\delta_{k,2}}{\sqrt{2D^2}}\right]\right\},\tag{6}$$

$k,l\in\{1,2\}$. $R^{(1)}$ can be obtained by inserting Eq. (5) into Eq. (2); similarly, $R^{(2)}$ can be obtained by inserting $\mathrm{Prob}(y^{(2)}_n=1)$—which results from the system of equations complementary to Eq. (3)—into Eq. (2). Evaluation of the SNR in the general case $\tau_1+\tau_2\neq T_s$ is more difficult since then the system (3) is not closed.

Theoretical curves SNR vs D show good agreement with numerical results, and the sensitivity of the SNR to changes of τ_1 and τ_2 for fixed w is correctly predicted (Fig. 1). In particular, by comparing the SNR from Eq. (2) for various combinations of time delays, the highest maximum SNR is observed for $w>0$ and $\Delta\phi_2=0,\pm2\pi,\pm4\pi,\ldots,$ [Fig. 1(a,b)], which yields optimum values of the delays $\tau_1=T_s-\Delta\phi/\omega_s$, $\tau_2=\Delta\phi/\omega_s$. Also for $w<0$ SR can be maximally enhanced if $\Delta\phi_2=\pm\pi,\pm3\pi,\pm5\pi,\ldots,$ [Fig. 1(c)], which is equivalent to choosing the optimum delays $\tau_1=T_s/2-\Delta\phi/\omega_s$, $\tau_2=T_s/2+\Delta\phi/\omega_s$. Hence, the above theory predicts the same optimum values of the time delays as deduced from the numerical simulations.

In this paper we have shown that in coupled threshold elements driven by periodic phase-shifted signals for fixed coupling strength SR can be enhanced by introducing optimum time delays in the coupling terms. The mechanism for this enhancement is the reduction of the effective phase shifts between periodic signals, perceived by each element, to the optimum values for which the SNR is maximally increased: for positive coupling, the elements are effectively driven in phase, while for negative coupling—in antiphase. The degree to which the SNR can be increased is thus limited by the maximum SNR in the two above mentioned cases. Anyway, for positive coupling, the maximum SNR can rise significantly in comparison with the case without delays depending on $\Delta\phi$. It should be pointed out that since we consider the SNR in individual elements, introduction of time delays not only increases the average performance of the two coupled units but also sensitivity of each element to the periodic signal at its own input. The presented method of the enhancement of SR is a kind of controlling SR in a system of coupled units with periodic signals shifted in phase. Generalization of the results of this paper to the case of spatially extended systems, e.g., chains of units with nearest-neighbor coupling driven by plane traveling waves [14] is straightforward. Thus, introduction of time delays in the coupling can also be a method to control SR in extended systems with spatiotemporal periodic signals. Time delays in coupling can naturally appear in many systems, e.g., in biological neural networks, in electric circuits as delays in transmission lines, etc. Our present results show that they can be of large importance for the detection of weak spatiotemporal periodic signals immersed in noisy background by means of SR.

A.K. is grateful to Darmstadt University of Technology for financial support and to H. Benner for his hospitality during a visit in Darmstadt in 2002. The authors are also grateful to H. Benner for helpful discussions and critical reading of the manuscript.

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